de-Rahm

$H^1 \xrightarrow{\nabla} H(\mathrm{curl}) \xrightarrow{\nabla\times} H(\mathrm{div}) \xrightarrow{\nabla\cdot} L^2 \to 0$

$S_h \to ND_h \to RT_h \to W_h \to 0$

$S_H \to ND_H \to RT_H \to W_H \to 0$

$S_H \to ND \to RT \to W_h \to 0$

$S_h \to Ed \to F \to E \to 0$

$V \to Ed \to$

$AV \leftarrow AEd \leftarrow AF \leftarrow AE$

How to construct

$W_H \to$ simply Restrict to AE the function we want to include in coarse space

RT_H : 1) Restrict to coarse faces the tangents.
2) Extension to interior

$$\min_{u \in RT(AE)} \|u\|^2_{H(\mathrm{div})}$$

s.t. $u|_{\partial AE}$ = tangent

$\mathrm{div}\, u \in W_H$

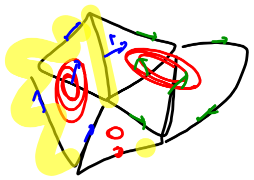
$u \in RT_h(AE)$
$p \in W_h(AE)$

$T = M_W \cdot$ tangents

$$\begin{bmatrix} M(+A) & B^T & 0 \\ B & 0 & T \\ 0 & T^T & 0 \end{bmatrix} \begin{bmatrix} u \\ p \\ \lambda \end{bmatrix} = \begin{bmatrix} M+A \text{ tang.} \\ B \text{ tang.} \\ 0 \end{bmatrix}$$

$M + A$: $(u,v) + (\mathrm{div}\,u, \mathrm{div}\,v)$

$u = u^{+} + \mathrm{curl}\,\varphi$, $\varphi \in H(\mathrm{curl})$



$P: RT_H \to RT_h$

$P(:,i) = \begin{bmatrix} 0 \\ x \\ x \\ x \\ x \\ x \\ x \\ x \end{bmatrix}$

3) ND_H 1) Restrict tangents to AEd.
2) Extend to AF
3) Extend to AE

AV: $S_H(AV)$

AEd: $S_H(AEd) \leftarrow ND_H(AEd)$

AF: $S_H(AF) \leftarrow ND_H(AF) \leftarrow RT_H(AF)$

AE: $S_H \leftarrow ND_H \leftarrow RT_H \leftarrow W_H$

Parallel in Time

$\vec{u}_i = \Phi_i(\vec{u}_{i-1}) + \vec{g}_i$

$\Phi_i = \Phi$

$\vec{u}_t - \kappa \Delta u = f$

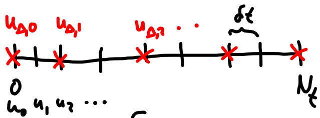
$M\vec{u}_i - \delta t \vec{f}_i = \vec{u}_{i-1}$ $\quad \Phi = M^{-1}$

$$M = \begin{bmatrix} & -\kappa\frac{\delta t}{(\Delta y)^2} & \\ -\frac{\kappa \delta t}{(\Delta x)^2} & 1+2\left(\frac{\kappa\delta t}{(\Delta x)^2}+\frac{\kappa \delta t}{(\Delta y)^2}\right) & -\frac{\kappa\delta t}{(\Delta x)^2} \\ & -\frac{\kappa\delta t}{(\Delta y)^2} & \end{bmatrix}$$

$\vec{u}_i = \Phi \vec{u}_{i-1} + \vec{g}_i$, $u_0 = u(0,\vec{x}) = \bar{u}_0$

$$A\vec{u} \equiv \begin{bmatrix} I & & & \\ -\Phi & I & & \\ & \ddots & \ddots & \\ & & -\Phi & I \end{bmatrix} \begin{bmatrix} \vec{u}_0 \\ \vec{u}_1 \\ \vdots \\ \vec{u}_{N_t} \end{bmatrix} = \begin{bmatrix} \vec{g}_0 \\ \vec{g}_1 \\ \vdots \\ \vec{g}_{N_t} \end{bmatrix} \equiv \vec{g}$$

$\vec{g}_0 \leftarrow \bar{u}_0$



C-points

m - coarsening factor (here: m=2)

$$A_\Delta \vec{u}_\Delta \equiv \begin{bmatrix} I & & & \\ -\Phi^m & I & & \\ & -\Phi^m & I & \\ & & \ddots & \ddots \end{bmatrix} \begin{bmatrix} u_{\Delta,0} \\ u_{\Delta,1} \\ \vdots \end{bmatrix} = \begin{bmatrix} g_{\Delta,0} \\ g_{\Delta,1} \\ \vdots \end{bmatrix}$$

P - ideal interp.

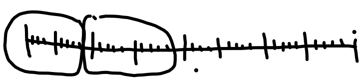
R - ideal restriction

$$\begin{bmatrix} I & & \\ -\Phi_\Delta & I & \\ & \ddots & \ddots \end{bmatrix}$$

exact method

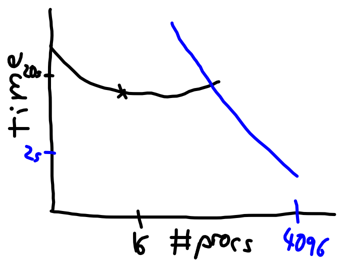
- F-relaxation
- CGC w/ A_Δ with Φ^m $\quad$ V-cycles

- CGC w/ A_Δ with Φ_Δ $\quad$ V-cycles
- FCF-relaxation



$$A = \begin{bmatrix} I & & \\ -\Phi & I & \\ & -\Phi & I \\ & & \ddots & \ddots \end{bmatrix} \qquad \hat{A} = \begin{bmatrix} I & & \\ -\hat{\Phi} & I & \\ & -\hat{\Phi} & I \\ & & \ddots & \ddots \end{bmatrix}$$

problem size $129^2 \times 16{,}384$



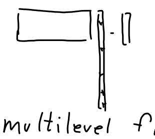
Minimization

$\min f(x)$; x^* is expected to be sparse

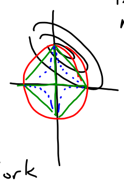
$\min f(x) + \mu\|x\|_1$

we've already worked on sparse modelling $m \sim 6$

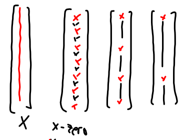
$\min \|Ax - b\|_2 + \mu\|x\|_1$



$\|x\|_0$ " $nnz(x)$



multilevel framework



x - fine level

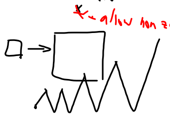
x_c - coarse level

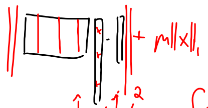
assumptions:

(1) relaxation on x_c

costs $\sim nnz(x_c)$

num of allowed non zeros

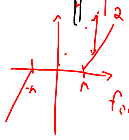




The main aim

if we choose the right relaxation (CD) & selection is good (S)

$\Rightarrow$ x remains sparse throughout the process



$\int_\mu GS(x)$

for sparse modelling $|A^T(\cancel{Ax_k} - b)|$

$$\begin{bmatrix} x_c \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} [x_c]$$

Covariance Selection

$\{x_i\}_{i=1}^k$ $x_i \in R^n$ $n \sim 10^6$

$k << n$ $k \sim 10^4$

$x_i \sim N(\bar{\mu}, \Sigma)$

$$P(x_i) \propto \frac{1}{(\det(\Sigma))^{1/2}} e^{-(x_i - \bar{\mu})^T \Sigma^{-1} (x_i - \mu)}$$

$\hat{\mu} = \frac{1}{k}\sum x_i$; $y_i = x_i - \hat{\mu}$ $(\Sigma^{-1})_{ij} = 0$

assumption: Σ^{-1} is sparse $(x)_i$ & $(x)_j$ - cond. independent

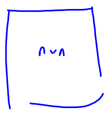
Maximum likelyhood

$$\min_{\substack{A > 0 \\ \text{symm.}}} f(A) = -\log\det(A) + \frac{1}{k}\sum_{i=1}^{k} y_i^T A y_i + \lambda\|A\|_1$$

$A^* \to \hat{\Sigma}^{-1}$

$\nabla f = -A^{-1} + Y = 0$ $Y = \frac{1}{N}\sum y_i y_i^T$

$\lim_{k \to \infty} \frac{1}{k}\sum y_i y_i^T = \Sigma$



A - sparse

A^{-1} - dense

1) efficient way to calc $\det(A)$ ← sparse

2) calc the entries of A^{-1} in specific locations:

most important : non-zeros of A

$P(A) \to P$: low order polynomial that approx. x^{-1}

simplification $A = D^T \Theta D \to$

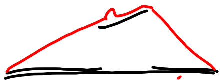
$D \sim \nabla_h$ Θ - diagonal & positive

$$\Delta \log\det(A) = \sum (A^{-1})_{ij} \Delta A_{ij}$$

$Ax = e_i \to \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \to i$ Jacobi + multigrid

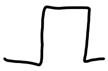
1 Vcycle in $O(\log n)$

AMR-RD



$Lu = f$

$u^0 = 0 \quad u' = u^0 + \delta u^0$



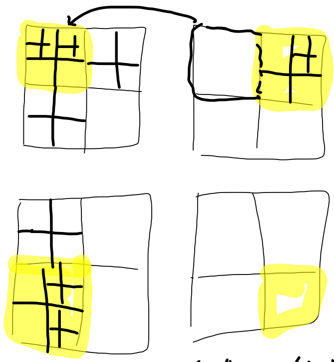
$L\delta u^0 = f \xrightarrow[RD]{Solve} L\delta u_i^0 = \chi_i f \quad \sum_{i=1}^{P} \chi_i = 1$

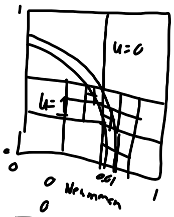
$\delta u^0 = \sum_i \delta u_i^0$

$u' = u^0 + \delta u^0$

Solve for $\delta u'$

$L\delta u_i' = \chi_i(f - Lu')$

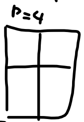




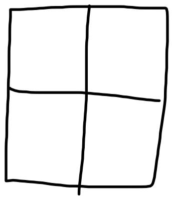
$\hat{u}(r) = \begin{cases} 1 \\ p(r) \\ 0 \end{cases}$

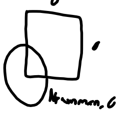
$Lu = f$

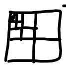
$P=4$



$\sin(kx)\sin(jx)$







E - # elements / proc possible

N_k - # Elems on Ω_k

$1 \le M_k \le M_0 \quad M_0 \sim O(1)$

$E = M_k N_k$

Results

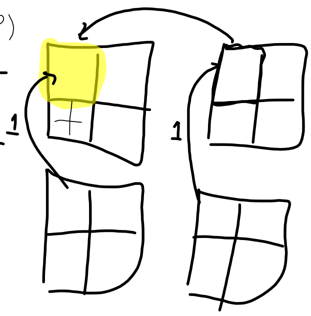
Sin Sin $P=128$

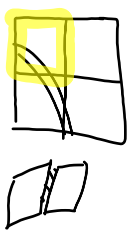
$M_0 \sim 2$

2 iterations

Comm. Pattern $\log_2(P)$

1	1	1	1
2	2	2	2
3	3	3	3
4	4	4	4
5	5	5	5
6	6	6	6
7	7	7	7
8	8	8	8



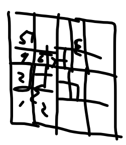


$\partial_{xx} + 0.01\partial_{yy} = f$

$-\Delta p = f$

$p = 0$

$r^\alpha \sin$



$P = 1{,}000$

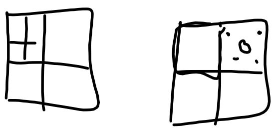
$\vec{U} = \nabla p$

$\nabla \cdot \vec{U} = f$

$\nabla \times \vec{U} = 0$

$+\|W(\nabla \cdot \vec{U} - f)\|^2$

$+\|W(\nabla \times \vec{U})\|^2$



$f - Lu^0$

$\langle L\phi_i, f - Lu^0 \rangle$

 +  =

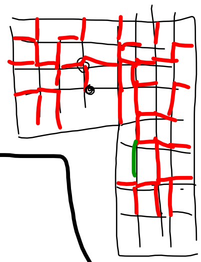
Randomized MG & FT

Basic observation: Deterministic ordering (sweeps) is artificial

what could we gain

- better numerical performance (RB)
- parallelism
- cache efficiency

Kelner (MIT) and digression



while $S \neq \emptyset$

pick $i \in S$

compute r_i = residual at x_i and $S = S \setminus \{i\}$

if $|r_i| > \tau$ then

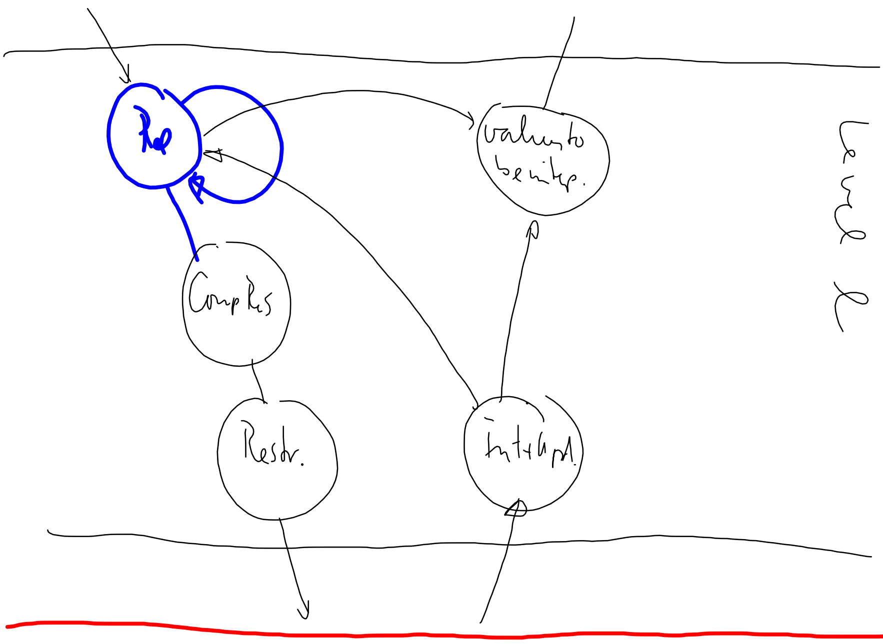
$x_i \mathrel{+}= r_i$

$S = S \cup N(x_i)$

endif

endwhile

The hard part: This must be done in a ML hierarchy!



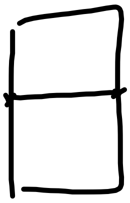
Why is this not done always and used everywhere?

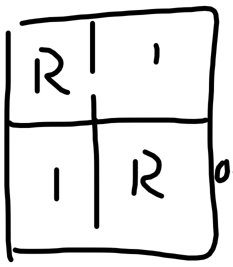
- the granularity of single unknowns is too small
- many details are architecture & machine dependent.

maybe future machines will support asynchronicity better?

Stokes

$$\begin{cases} -\mathrm{div}\, \sigma = \vec{f} \text{ in } \Omega \\ \mathrm{div}\, \vec{u} = 0 \end{cases}$$



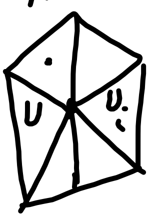


$H^{1+\alpha}(\Omega)$

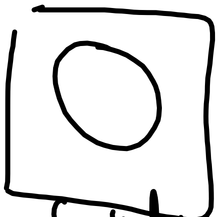
$0<\alpha\ll 1$

$u_i = \fint_K u\,dx$

$$\sigma = 2\underline{\nu}\,\mathcal{E}(\vec{u}) + p\,\underline{I}$$



$$\mathcal{E}(\vec{u}) = \frac{1}{2}\left(\nabla\vec{u} + (\nabla\vec{u})^T\right)$$



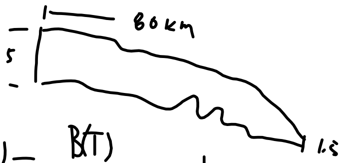
$$Iu = \sum u(z_i)\varphi_i(x)$$

$$Iu = \sum u_i \varphi_i(x) \qquad u_i = \frac{\int_{V_i} u\,dx}{|V_i|}$$

$$\|\ (u - Iu)\|_{0,\underline{K}} \leq C h_K \|\nu\nabla u\|_{0,\omega_K}$$

$$\|\nu\nabla(Iu)\|_{0,K} \leq C\,\|\nabla u\|_{0,\omega_K}$$

Glacier Problem



$\underline{u}$,

p

T

$U = \nabla u$

$\mathcal{E} = U + U^T$

$\mathcal{E}_e = \|\mathcal{E}\|_F$

$\nu = \dfrac{B(T)}{\mathcal{E}_e^{2/3}} = \dfrac{1}{\mu}$

$\sigma' = \nu \mathcal{E}$

$\sigma = \nu \mathcal{E} - pI$

$$\frac{\partial u}{\partial t} = \nabla \cdot \sigma' - \nabla p + f$$

$$\nabla \cdot \underline{u} = 0$$

FOSLize

$\nabla \cdot \underline{u} = 0$

$\underline{U} = \nabla \underline{u}$

$$\frac{\partial \underline{u}}{\partial t} - \nabla \cdot \nu (U + U^T) + \nabla p = f$$

$\nabla \times U = 0$

$\nabla \cdot U^T = 0 \,/\, \nabla tr U = 0$

$$E = \begin{matrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{matrix}$$

$\mathcal{E} = EU$

$\mathcal{E}_e^2 = \langle EU, EU \rangle$

$M = \dfrac{1}{\nu}$

$\hat{U} = \nu U$

$Lu = F$

$$\begin{bmatrix} 0\;0\;0\;0 & \partial_x \partial_y & 0 \\ M\;0\;0\;0 & -\nabla & \\ \quad M & & \\ \quad\quad M & -\nabla & \\ \quad\quad\quad M & & \\ \partial_x\;\partial_y\;\partial_y\;0 & 0 & -\nabla \\ 0\;\partial_x\;\partial_x\partial_y & & \\ -\nabla\times M & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{U}_{11} \\ \hat{U}_{12} \\ \hat{U}_{21} \\ \hat{U}_{22} \\ u_1 \\ u_2 \\ p \end{bmatrix} = f$$

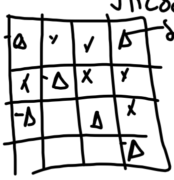
$(L^* L)$

$\rho = .96$

$\rho = .7$

Unknown Based

JR code



No Stress

ACE

Robin

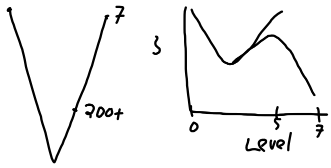
$u + \alpha \dfrac{\partial u}{\partial n} = g$

$$-\Delta \otimes \begin{bmatrix} 1 & 1-\varepsilon \\ 1-\varepsilon & 1 \end{bmatrix}$$

$$\boxed{\begin{matrix} -\Delta & & 2\partial_{xy} \\ & -\Delta & \partial_{xx} \\ 2\partial_{xy} & \partial_{xx} & -\Delta \end{matrix}}$$

Non Galerkin Coarse Grids

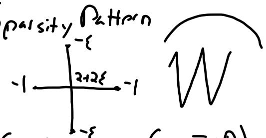
P^TAP



$P^TAP = A_g$

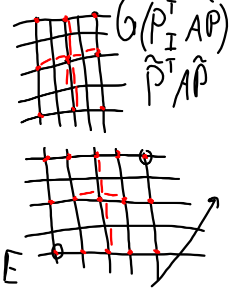
$A_c \approx A_g$

① Sparsity Pattern



$G(P_I^T A P) \cup G(P^T A P_I)$

$\hat{P}^T A \hat{P}$



$A_c = A_g + E$

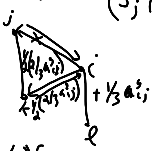
$\|e_i\| \le \gamma \|a_i^g\|$

② Find entries

Form A_g, $\mathcal{N}$ the sparsity pattern

$a_{ij}^g \notin \mathcal{N}$

$(S_j \cap \mathcal{N}_i) = \{k, \ell\}$



$G(A_c) \in \mathcal{N}$

$A_c \leftarrow \frac{1}{2}(A_c + A_c^T)$

$$\| I - A_g^{-1} A_c \|_2 \le \theta < 1$$

$$E_{PTG} \le E_{TG} \frac{1}{1 \pm \theta}$$

P, R, P_0, R_0

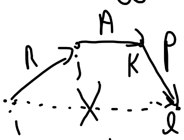
$A_g = RAP$ $S_p(A_g) = S_p(R_0 A P_0)$

$S_p(P_0) \subset S_p(P)$ $P_0 =$ tent. Agg

$I - \cancel{P_0}(R_0 A P_0)^{-1} R_0 A$

$A_c = \cancel{A} R_0 A P_0$

$(A_g)_{ij}$



$P = (I - \omega D^{-1} A) P_0$

$A_g 1 = A_c 1$

$1^T A_g = 1^T A_c$

$\frac{1}{2} P_0^T A P_0$

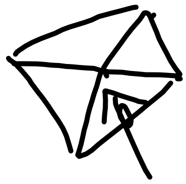
$$\text{error} \sim \left(\frac{e_j}{e_i} + \frac{e_k}{e_i} - 1\right)$$

$$\left(I - \begin{bmatrix} & -3 & \\ -3 & 12 & -3 \\ & -3 & \end{bmatrix}^{-1} \frac{1}{3}\begin{bmatrix} -1 & -1 & -1 \\ -1 & 8 & -1 \\ -1 & -1 & -1 \end{bmatrix} \right)$$

$$\begin{pmatrix} & -1 & \\ -1 & 4 & -1 \\ & -1 & \end{pmatrix}$$

Cloth

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad x_1 = \begin{pmatrix} x_1^1 \\ x_1^2 \\ x_1^3 \end{pmatrix}$$



$$F = m\ddot{x}$$

$$\begin{pmatrix} \Delta x \\ \Delta v \end{pmatrix} = h \begin{pmatrix} V_0 + \Delta V \\ M^{-1} f(x_0 + \Delta x,\ V_0 + \Delta V) \end{pmatrix}$$

$$\left(I - hM^{-1}\frac{\partial f}{\partial v} - h^2 M^{-1}\frac{\partial f}{\partial x}\right)\Delta v = hM^{-1}\left(f_0 + h\frac{\partial f}{\partial x}V_0\right)$$

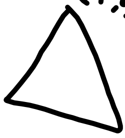
m_1 m_1 m_1

$$\left(M - h\frac{\partial f}{\partial v} - h^2\frac{\partial f}{\partial x}\right) \qquad C(x) = 0$$

$$\oint f\, ds = 0 \Rightarrow \nabla \times f = 0 \qquad E(x) = \frac{K}{2} C(x)^T C(x)$$

$$f = -\nabla E$$

(x,y,z)

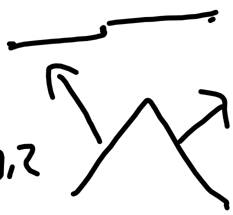


(u,v)

$w(u,v) \to x,y,z$

$w_u(x)$

$w_v(x)$



LSA

SA Relaxation

$Ax=0$

$Ay=0$

$\vee$

$S+\theta$

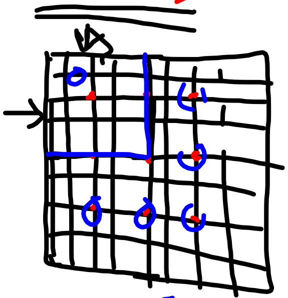
$S-\theta$

$(XY)^T A (XY)$

$(XY)^T (XY) = I$

$(XY)Q$

MLDD



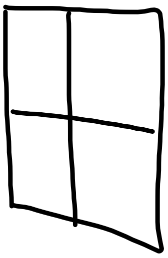
N_p

$x^{(ai)}$ Domain

$A_i = A$ $\quad \sum Q_i = I$

δx_i $\quad A_i x_i = r_i = r$

$x^{n+1} = x^n + \sum Q_i \delta x_i$



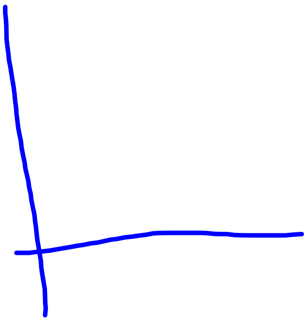
$$P_h^{2h} = \begin{bmatrix} P_{00} & P_{01} \\ P_{10} & P_{11} \end{bmatrix}$$

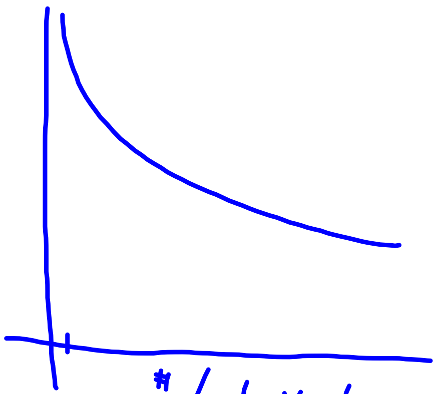
$$P^r \begin{bmatrix} I & 0 \\ 0 & P_{11} \end{bmatrix} A_c^{-1} (P^c)^T A P^c$$

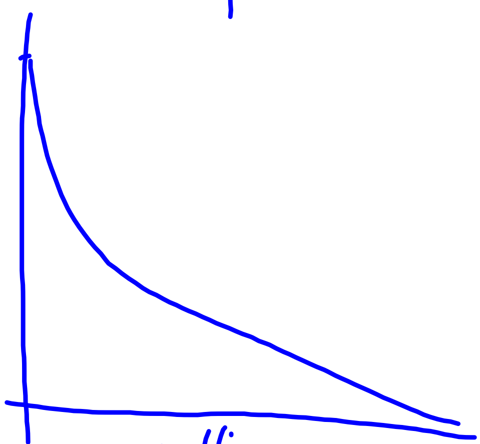
Range

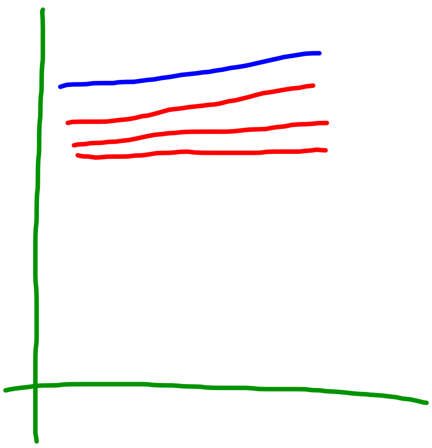
$A_i x_i = Q_i r$

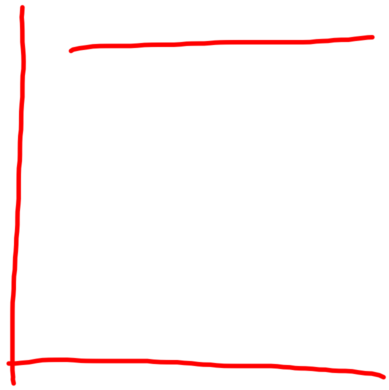
$x^{n+1} = x^n + x_i$

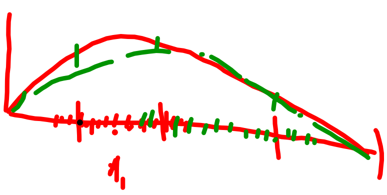


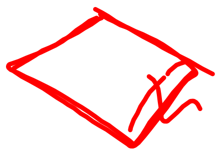












Strong Disturbances

subsurface flow → Darcy's law

Groundwater: surface

Reservoir simulation: wells, fractions

Balance equations (per phase / per component)

$\mu \frac{\partial S_r}{\partial t} - \nabla K \nabla p + q = 0$ → source term

↳ Newton $J = \begin{pmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{pmatrix}$

→ CPR: 1) roughly "approx. pressure: $A_{pp}^{-1} \approx A_p^{-1}$!
2) ILU for the full system

A_{pp} may have unwanted structures:
- well equations: not really a problem
- source terms: cells are perforated
  ⇒ influences the diagonal A_{pp}

⇒ - weak diagonal dom. lost
- indefinite A_{pp}
- strong divergence

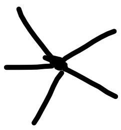
What to do:

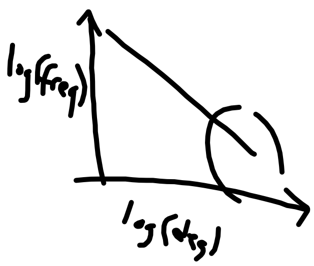
1) adapt AMG
2) keep problems away from AMG ⇒ Schur 2
3) Keep the problems away from A_{pp}

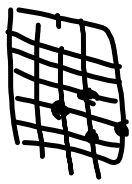
$$J = \begin{pmatrix} A_{pp} & A_{ps} \\ A_{sp} & A_{ss} \end{pmatrix}$$

Scaly Graphs

$Lx = b$





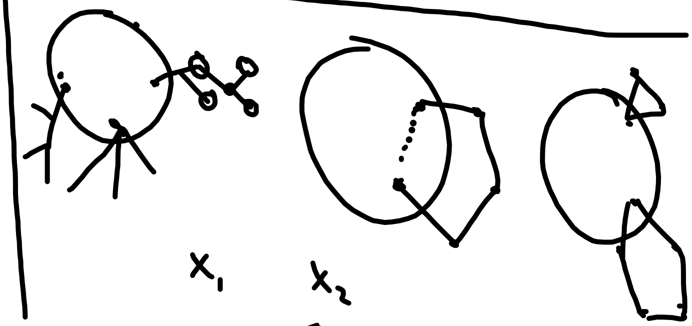


LAMG : (Livne/Brandt)

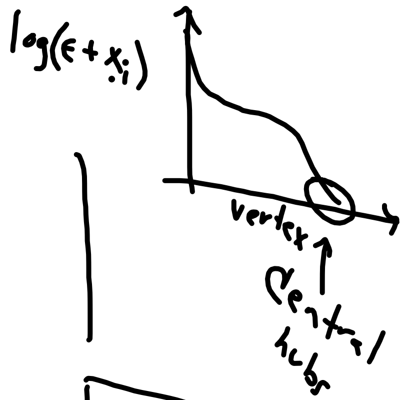
- Elimination
- Aggregation

$\deg = (1, 2, 3), 4$

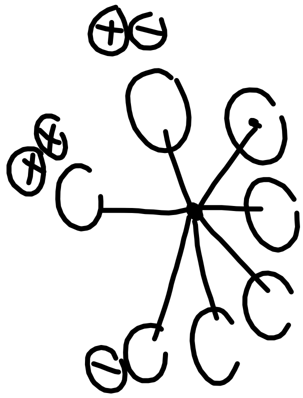
V W

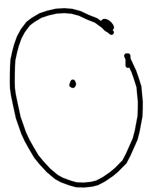


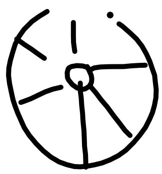
High-Degree Vertices



Smooth x







$\boxed{\text{WLOG}}$ $x \neq 0$, Steve!

$$|x_i|^2 \le \|x - P x_c\|^2 \le \frac{C\langle Lx, x\rangle}{\|L\|}$$

$$\sum_{i \in S} |x_i|^2 \le \frac{C\langle Lx, x\rangle}{(1)^S \|L\|}$$

Model Order Reduction (MOR)

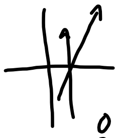
Full Approximation Scheme (FAS)

$u \, u_x$

Anisotropic Diffusion

$[W\ I]^T$ $[0\ I]$

$Q = PR$



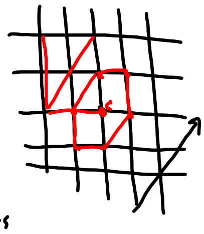
P_{ca} $\eta \Leftarrow \max_{\|x\|=1} \frac{\langle A Q_x, Q_x \rangle}{\langle Ax, x \rangle}$

0

$\pi/16$

$2\pi/16$

Q1



$P = \begin{bmatrix} W \\ I \end{bmatrix} \begin{matrix} \text{Fpts} \\ \text{Cpts} \end{matrix}$

$\hat{P} = \begin{bmatrix} \vdots & & \\ \vdots & \vdots & \\ & \vdots & \ddots \end{bmatrix}$

$G(C^k \hat{P})$

$k = 1, 2, 3, 4$

C6 so solve

$A P_j = 0$ $j = 1, \ldots, n_c$

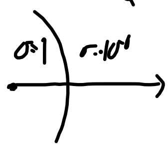
$\ni P\mathbb{1}_c = \mathbb{1}_F$ $P \in \mathcal{N}$

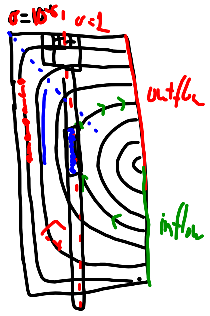
$\mathcal{N} = G(C^k \hat{P})$

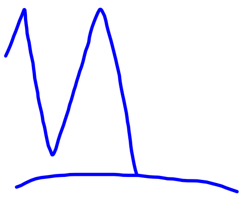
$\theta \backslash k$	1	2	3	$\boxed{4}$
$\pi/16$	7	5	3	1.2.. SA is scalable
$2\pi/16$				
0	1.0..			
$4\pi/16$	1.~			

FOSL-ize $B \cdot \nabla u + \sigma u = f$

$\nabla \cdot \underbrace{B^T B}_{2\times 2} \nabla u$





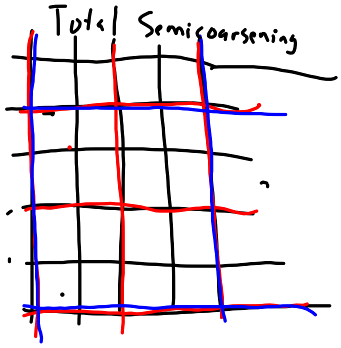


① Smooth P w/ $\begin{bmatrix} W \\ I \end{bmatrix}$ $[W]$ $\to$ $P \approx \begin{bmatrix} A_{FF}^{-1} A_{FC} \\ I \end{bmatrix}$ A_{FF}^{-1} "easy"?

② Minimum Caliber?
2, 3, 4?

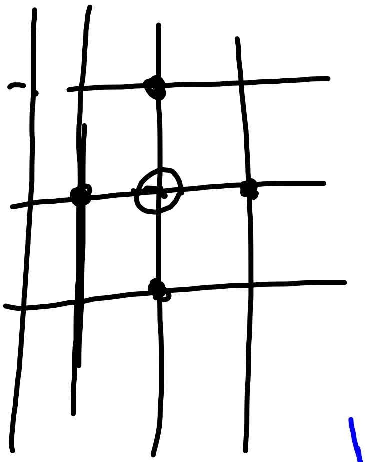
③ Adaptivity needed for best iteration counts

Total Semicoarsening



Under the rug ... strength of connection /
Algebraic distance

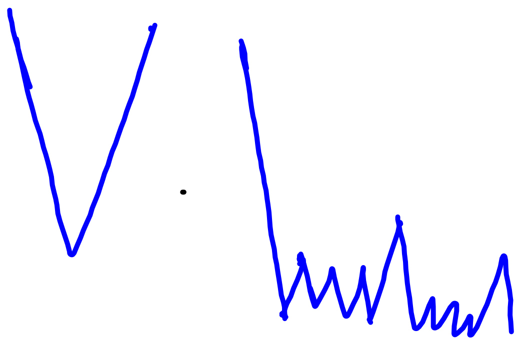
Uncertain Diffusion



$$E(u) = \int k(x)\left(\nabla u(x)\right)^2 dx$$

$$P(u) \sim e^{-E(u)/T}$$

$$O(\varepsilon^{-1/2})$$



FAS

$$L^{2h}(u^{2h}) = L^{2h}(I_h^{2h} u^h) - I_h^{2h}(L^h(u^h)) = \tau_h^{2h}$$

[Problem: $L^h(u^h) = 0$.]

$Lu = u'$

FOSLS $\quad Lu = f \quad \|Lu - f\|^2 \quad \dfrac{\|u' - u^{*\prime}\|^2}{\|Lu\|^2}$

Helmholtz

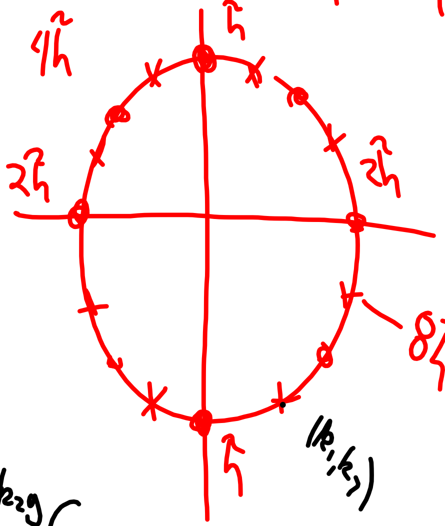
$+\Delta p + K^2 p$

$K = (k_1, k_2)$

$K^2 = k_1^2 + k_2^2$

$|K| h < \frac{1}{8}$ $4\tilde{h}$

$e^{k\pi \cdot x}$



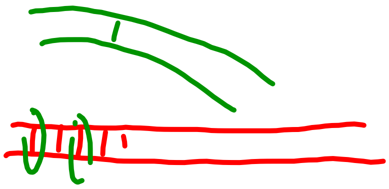
$e^{k_1 x + k_2 y} \int =$

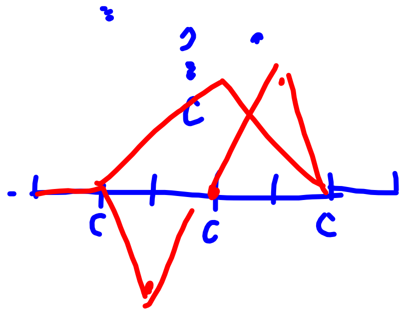
Systems

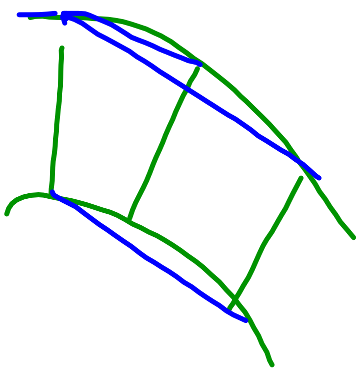
$$\overset{SA}{\begin{pmatrix} P_f \\ P_c \end{pmatrix}} \rightarrow \overset{AMG}{\begin{pmatrix} W \\ I \end{pmatrix}} \qquad P_f P_c^{-1}$$

$$b - PIb$$

$$P^T \begin{bmatrix} A_{11} & 0 & A_{13} \\ 0 & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} P_1 & x & y \\ x & P_2 & x \\ x & y & P_3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \qquad 1$$







FOSLS/FOSLL*

FOSLS 3 Examples

1.) $-\Delta p = f \ \Omega$ I $\underline{U} - \nabla p = 0$

$p = 0 \ \delta\Omega$ $\nabla \cdot \underline{U} = -f$

$\nabla \times U = 0$

$L\mathcal{U} = F \quad L = \begin{bmatrix} I & -\nabla \\ \nabla\cdot & 0 \\ \nabla\times & \end{bmatrix} \begin{pmatrix} U \\ p \end{pmatrix} = \begin{pmatrix} 0 \\ -f \\ 0 \end{pmatrix}$

Min

$\|L\mathcal{U}^h - F\|^2$

$\langle L\mathcal{U}^h, L\mathcal{V}^h \rangle = \langle F, L\mathcal{V}^h \rangle$

Formal Normal

$L^* L$

$\begin{bmatrix} I & -\nabla \\ \nabla\cdot & 0 \end{bmatrix} \begin{bmatrix} I & -\nabla \\ \nabla\cdot & 0 \end{bmatrix}$

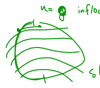
$\left\langle \begin{bmatrix} (I - \nabla\nabla\cdot) & -\cancel{\nabla} \\ \cancel{\nabla}\cdot & -\Delta \end{bmatrix} \mathcal{U}, \mathcal{U} \right\rangle$

$\langle \underline{u}, \underline{u} \rangle + \langle \nabla\cdot\underline{u}, \nabla\cdot\underline{u} \rangle + \langle \nabla p, \nabla p \rangle$
$+ \langle \nabla\times u, \nabla\times u \rangle$

$\begin{bmatrix} I - \nabla\nabla\cdot + \nabla\times\nabla\times & -\nabla \\ \nabla\cdot & -\Delta \end{bmatrix}$

② $b\cdot\nabla u = f$ min $\|b\cdot\nabla u^h - f\|^2$

$u = g$ inflow



$L = b\cdot\nabla$

$L^* L = -\nabla\cdot(bb^T)\nabla$

$c_0\|u\| \le \|b\cdot\nabla u\|$

$\|u\| + \|b\cdot\nabla u\|$

3) Stokes

$\frac{1}{\nu}\Delta\underline{u} - \nabla p = f$ ① $U = \nabla\underline{u}^T = [\nabla u_1, \nabla u_2]$

$\nabla\cdot\underline{u} = 0$ ② $\nabla\cdot U - \nabla p = f$

$\nabla\cdot\underline{u} = 0$

$\nabla\times U = 0$

$\nabla \operatorname{tr} U = 0$

$L^* L = \begin{bmatrix} -\Delta & & x \\ & \ddots & x \\ x \ x & y \ y & -\Delta \end{bmatrix}$

FOSLL*

$L^* L \sim H^2$ regular

$c_0\|\mathcal{U}\|_2 \le \|L^* L\mathcal{U}\| \le c_1\|\mathcal{U}\|_2$

$\|L\mathcal{U}^h - F\|_2 = \|Le^h\|_{?}$

FOSLL*

$L\mathcal{U} = F \qquad \langle L^*\mathcal{W}^h, L^* z^h \rangle = \langle F, z^h \rangle$

$LL^*\mathcal{W} = F$ GP

$\|L^*\mathcal{W}^h - \mathcal{U}\|$

$\langle L^*\mathcal{W}^h, L^* z^h \rangle = \langle \mathcal{U}, L^* z^h \rangle$

$= \langle L\mathcal{U}, z^h \rangle$

$\mathcal{U}^h = L^*\mathcal{W}^h \qquad = \langle F, z^h \rangle$

(1) FOSLS $\|L\mathcal{U}^h - F\|$

1   D

$c_0\|\mathcal{U}\|_1 \le \|L\mathcal{U}\| \le c_1\|\mathcal{U}\|_1$

$c_0 = \frac{1}{D^3}$

$\|e^h\| = O(h^{p+1})$

$\|Le^h\| = O(h^p)$

$\|U^h - \nabla p^h\|$

DOF

Hybrid FOS LS/LL*

$\mathcal{H}(\mathcal{U}, \mathcal{W}; F, \mathcal{U}) = \|L\mathcal{U}^h - F\|^2 + \|L^*\mathcal{W}^h - \mathcal{U}^h\|$

$\|L^*\mathcal{W}^h - \mathcal{U}\|^2$

$\begin{bmatrix} L^* & 0 \\ L^* & -I \\ 0 & L \end{bmatrix} \begin{pmatrix} \mathcal{W}^h \\ \mathcal{U}^h \end{pmatrix} = \begin{pmatrix} \mathcal{U} \\ 0 \\ F \end{pmatrix}$

$\mathcal{L}\xi = \mathcal{F}$

$\begin{bmatrix} L & L & 0 \\ 0 & -I & L^* \end{bmatrix} \begin{bmatrix} L^* & 0 \\ L^* & -I \\ & \end{bmatrix} = c_0 \boxed{\begin{matrix} 2LL^* & -L \\ -L^* & L^*L + I \end{matrix}}$

$4\|L^*(\mathcal{W}^h - \mathcal{W})\|^2 + \|\mathcal{U}^h - \mathcal{U}\| + \|L(\mathcal{U}^h - \mathcal{U})\|^2$

$\|e^h\|^2 + \|Le^h\|^2$

$L: \mathcal{V}_1 \to \mathcal{V}_2$

$\|Lu\|_{\mathcal{V}_2} + \|\mathcal{U}\|^2_{\mathcal{V}_1}$

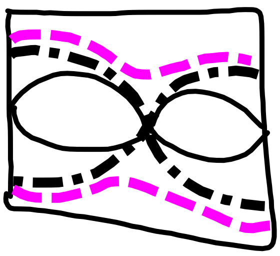
$b\cdot\nabla u = f$

FOSLL*

Hybrid $\|u\|^2 + \|b\cdot\nabla u\|^2$

Liquid Crystals

Lesdin



$$\frac{\partial u}{\partial t} + \nabla p - \frac{1}{Re}\nabla^2 u + j \times B = f$$

$$\to \nabla \cdot u = 0$$

$$\frac{\partial B}{\partial t} - \nabla \times (u \times B) - \frac{1}{S}\nabla \times \nabla \times B = g$$

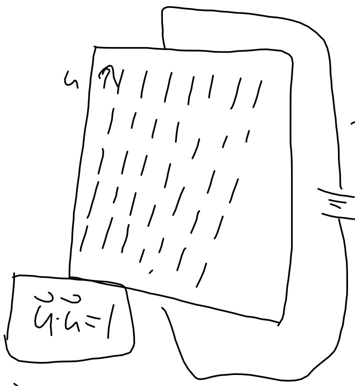
$$\to \nabla \cdot B = 0$$

$Re \approx 25{,}000$

$Re \doteq 50{,}001$

$\rho = 0.98$

~ 0.84



$\vec{u}\cdot\vec{u} - 1 = 0$

$u \cdot u = 1$

$$\frac{1}{2}\int k_1|\nabla\cdot u|^2 + k_2|u\cdot\nabla\times u|^2 + k_3|u\times\nabla\times u|^2 - \vec{D}\cdot\vec{E}$$

$D(u, E)$

$$= \frac{1}{2}\ldots \qquad A = I - \left(1 - \frac{k_2}{k_3}\right)uu^T$$

$$\boxed{\|\nabla\cdot u\|^2 + k_3\langle A\nabla\times u, \nabla\times u\rangle}$$

$$\begin{pmatrix} A_{uu} & A_{u\phi} & A_{u\lambda} \\ A_{\phi u} & A_{\phi\phi} & 0 \\ A_{\lambda u} & 0 & 0 \end{pmatrix}$$

$\boxed{n = u}$

u

ϕ

$\phi\cdot\phi = 1$

$\cdot\nabla\phi$

$$D = \varepsilon_1 E + \varepsilon_2 (n\cdot E)\, n\cdot E$$
$$+ \varepsilon_3 n(\nabla\cdot n) + \varepsilon_3 n\times(\nabla\times n)$$

Preconditioners

SA $(I - \omega D^{-1} A)$

Emin $\begin{cases} AP = 0 \\ \text{constrained} \\ PB_c = B \end{cases}$ ← CG method

agg coarsening
K CG iter

Newton's method

SA	Emin(5)	Emin(5,1)
30	14 →	